BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW OF MATHEMATICAL SCIENCE

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BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY.

THE SEATTLE MEETING OF THE SAN FRANCISCO SECTION.

A SPECIAL meeting of the San Francisco Section of the American Mathematical Society was held at Seattle, on Thursday and Friday mornings, June 17–18, in connection with the meetings of the Pacific division of the American Association for the Advancement of Science. Professor Blichfeldt, chairman of the Section, presided at the Thursday morning session, Professor Moritz temporarily filling the chair during the presentation of Professor Blichfeldt's paper. At the Friday morning session Professor Moritz presided. In the absence of Professor Bernstein, Professor Bell acted as temporary secretary.

The attendance included the following twelve members of the Society:

Professor E. T. Bell, Professor H. F. Blichfeldt, Professor A. F. Carpenter, Professor E. C. Colpitts, Professor G. I. Gavett, Dr. G. F. McEwen, Professor W. E. Milne, Professor R. E. Moritz, Professor L. I. Neikirk, Dr. L. L. Smail, Dr. A. R. Williams, Professor R. M. Winger.

The following papers were presented at this meeting:

(1) Professor W. F. DURAND: "Some points of contact between mathematics and applied science."

(2) Professor H. F. BLICHFELDT: "On the approximate representation of irrational numbers, and the theory of geometry of numbers."

(3) Professor E. T. BELL: "An arithmetical dual of Kummer's quartic surface."

(4) Dr. T. H. GRONWALL: "On the Minkowski volume and surface theory."

(5) Professor A. F. CARPENTER: "Congruences associated with a ruled surface."

(6) Professor FLORIAN CAJORI: "Moritz Cantor, the historian of mathematics."

(7) Professor L. I. NEIKIRK: "The functional variable. Second and higher derivatives."

(8) Dr. L. L. SMAIL: "Notes on some points in the theory of summable series."

(9) Dr. L. L. SMAIL: "Bibliography of the theory of summable series."

(10) Professor E. T. BELL: "An image in four dimensional lattice space of the theory of the elliptic theta functions."

(11) Professor E. T. BELL: "Certain arithmetical consequences of the equation of three terms in elliptic functions."

(12) Professor E. T. BELL: "Systems of higher determinate equations."

(13) Professor W. E. MILNE: "Infinite systems of vectors."

Professor Durand was introduced by Professor Blichfeldt. In the absence of the authors, Dr. Gronwall's paper was read by title and Professor Cajori's was read by Professor Blichfeldt. Professor Bell's last three papers were also read by title. Abstracts of the papers follow below.

1. Professor Durand's paper first discusses the types of problem that arise in applied science with especial reference to their mathematical requirements, and suggests certain lines for the classification of such problems. Following this general view of the subject, mention is made of some specific problems for the treatment of which adequate mathematical means are still lacking. This section of the paper is intended to suggest to mathematicians needs for the further development of mathematical methods and means of analysis in order to treat adequately actual problems that arise in various fields of applied science.

2. Professor Blichfeldt discusses certain fundamental theorems in the geometry of numbers, after which the approximate solution in integers of a system of linear equations is taken up, in particular of the system $x_i + a_i - x_1 x_{n+1} + b_i = 0$ $(i = 1, 2, \cdots, n)$. The results of Kronecker, Hermite and Minkowski are compared, and a recent theorem on this system by the author is stated.

3. If E is a homogeneous algebraic equation in x, y, z, w, it represents a surface S when the variables are point coordi-

nates; if the variables are interpreted arithmetically, E can be made to represent a system S' of theorems concerning integers. When S implies S' and S' implies S, Professor Bell calls S, S' arithmetical duals of each other, and determines an S' for Kummer's S, such that all the geometrical properties of S imply and are implied by systems of arithmetical theorems, which in turn may be interpreted in terms of lattice configurations lying upon two concentric spheres in space of sixteen dimensions.

5. Paul Serret* has proved the theorem "the double-ratio of the four points in which any generator of a ruled surface intersects four fixed asymptotic curves is constant." Analogously, any four directrix curves of a ruled surface which cut all the generators in four points of constant double-ratio may be called a Serret set. Let C_y, C_z, C_μ, C_ν be any four directrix curves of a ruled surface S, P_y, P_z, P_μ, P_ν the points where they cut any generator g, and T_y, T_z, T_μ, T_ν the tangents to C_y, C_z, C_μ, C_ν at these points. T_y, T_z, T_μ determine a quadric $K\theta$. Professor Carpenter's paper proves that T_ν will lie upon $K\theta$ and be a ruling of the same kind as T_y, T_z, T_μ if and only if C_y, C_z, C_μ, C_ν constitute a Serret set, and that there is a one-parameter family of such quadrics $K\theta$ associated with each generator g of S. The generator g belongs to one regulus of each quadric $K\theta$. The paper next considers one of the quadrics $K\theta$ whose equation is invariant in form. One of these quadrics is associated with each generator of S, and this family of quadrics gives rise to two congruences Δ_1 and Δ_2 whose properties are investigated.

6. Professor Cajori gives a biographical sketch of Moritz Cantor and an estimate of his work as a historian.

7. Volterra and others have given implicit definitions for a function of a line and its first derivative. Professor Neikirk, following the method of one of his previous papers, gives explicit definitions of second and higher derivatives. Among other results, he gives the following classification of functions of a functional variable: (1) The first derivative is a point function. The second derivatives are zero or infinite. For functions of this class the methods of this paper may be

* P. Serret, Théorie nouvelle géométrique et mécanique des Lignes à double Courbure, Paris, Bachelier, 1860.

used to derive the second variation as used in the calculus of variations. (2) The first derivative is a function of a functional variable. The second derivatives are the results of repeated operations. (3) The first derivative is a mixed, point and functional variable, function. The second derivatives are the results of the repeated operations except for the case F_{ii}, which is infinite.

8. In Dr. Smail's first paper, a number of minor gaps in the systematic treatment of the theory of summable divergent series are filled in, and new proofs of some familiar theorems are given.

9. Dr. Smail's second paper gives a bibliography of the theory of summable series.

10. In this paper Professor Bell proves eleven geometrical theorems concerning point configurations upon a sphere in lattice space of four dimensions which imply, and are implied by, the theory of the elliptic theta functions.

11. Extending Liouville's theorems in the fifth of his memoirs on general formulas useful in the theory of numbers, Professor Bell shows that similar general formulas exist for a pair of bilinear forms each in four variables, and finds a complete set of 57 such formulas.

12. Professor Bell shows in this paper by two detailed examples how the general formulas of the kind given in the preceding paper may be used to derive information concerning the number of solutions of certain systems of indeterminate equations of any degree.

13. In this paper Professor Milne shows that corresponding to every infinite set of vectors of finite norm in infinitely many dimensions there exists an infinite set of constants λ_i ($i = 1, 2, \cdots$) with the following properties: The necessary and sufficient condition that (*a*) the system be orthogonal is that $\lambda_i = 1$ for every i; (*b*) the system have an adjoint is that $\lambda_i > 0$ for every i; (*c*) the system be essentially linearly dependent is that $\lambda_i = 0$ for some i. Applications are made to infinite matrices and to infinite sets of linear equations.

E. T. BELL,
Acting Secretary of the Section.

NOTE ON A GENERALIZATION OF A THEOREM OF BAIRE.

BY PROFESSOR E. W. CHITTENDEN.

(Read before the American Mathematical Society September 8, 1920.)

IT is the purpose of this note to call attention to the following generalization of a celebrated theorem of Baire:

THEOREM: *Let P be a perfect set in a complete, separable, metric space S.* Then a necessary and sufficient condition that a function defined on P be of class 1 in the Baire classification of functions is that the function be at most point-wise discontinuous on every perfect subset of P.*

The necessity of the condition has been established by Hausdorff.†

A satisfactory proof of this theorem can be obtained from a proof given by Vallée-Poussin‡ for the corresponding theorem in space of n dimensions by making appropriate changes in terminology, since Vallée-Poussin makes no essential use of the special properties of space of n dimensions in his argument. The methods to be followed in generalizing this proof of Vallée-Poussin are sufficiently indicated in the treatise of Hausdorff already cited and in the memoirs of Fréchet.§

There is, however, one point which requires special consideration. Vallée-Poussin introduces an auxiliary function with the following definition: If M is a point of S; H_1, H_2, $\cdots$, H_n are closed sets such that no two have a common element; δ_1, δ_2, $\cdots$, δ_n are the respective distances of H_1, H_2, $\cdots$, H_n from M; and a_1, a_2, $\cdots$, a_n are constants; then

* See Hausdorff, Grundzüge der Mengenlehre, Veit und Co., Leipzig, 1914, pp. 211, 315. The word "vollständig" is here translated "complete." A complete set admits a generalization of the theorem of Cauchy in the sense of Fréchet, "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matem. di Palermo*, vol. 22 (1906), pp. 1–74.

† Hausdorff, loc. cit., p. 388.

‡ Intégrales de Lebesgue, Fonctions d'Ensemble, Classes de Baire, Gauthier-Villars, Paris, 1916, pp. 105–125.

§ As an example of the application of the theory of transfinite ordinal numbers in the present situation, see Fréchet, "Les ensembles abstraits et le calcul fonctionnel," *Rendiconti del Circolo Matem. di Palermo*, vol. 30 (1910), pp. 5–10.

$$\varphi(M) = \frac{a_1/\delta_1 + a_2/\delta_2 + \cdots + a_n/\delta_n}{1/\delta_1 + 1/\delta_2 + \cdots + 1/\delta_n},$$

except when M is in H_i; then $\varphi(M) = a_i$ $(i = 1, 2, 3, \cdots, n)$.*

The function $\varphi(M)$ is continuous in space of n dimensions. I wish to show that it is continuous in the space S. It is sufficient to show that the distance δ of a point M from a closed set H is a continuous function of M, vanishing on H.

Let (A, B) denote the distance between two points A, B of the metric space S. Then we have the following fundamental property of distance: If A, B, C are any three points of S,

$$(A, B) \leqq (A, C) + (C, B).\dagger$$

If H is a closed set the distance (M, X), where X is in H, has a minimum δ at a point X_0. We call $\delta = (M, X_0)$ the distance from M to H. Let M' be any other point of H, and $\delta' = (M', X_0')$ be the corresponding distance from H. Then we have

$$\delta = (M, X_0) \leqq (M, X_0') \leqq (M, M') + (M', X_0')$$
$$\leqq (M, M') + \delta'.$$

Similarly, $\delta' \leqq \delta + (M, M')$. Therefore $|\delta - \delta'| \leqq 2(M, M')$, and the continuity of δ is established.

UNIVERSITY OF IOWA,
IOWA CITY, IOWA.
February 28, 1920.

CERTAIN ITERATIVE CHARACTERISTICS OF BILINEAR OPERATIONS.

BY DR. NORBERT WIENER.

Introduction.

IN a recent paper‡ the author has developed the necessary and sufficient condition that a bilinear operation in two variables should generate by iteration all rational operations with

* Vallée-Poussin, loc. cit., p. 118.

† Hausdorff, loc. cit., p. 211. See also Fréchet, "Les notions de limite et de distance," *Transactions Amer. Math. Society*, vol. 19 (1918), p. 54.

‡ *Annals of Mathematics*, vol. 21, No. 3 (March, 1920), pp. 157–165.

rational coefficients. This is a particular case of the general problem of determining just what operations any given bilinear operation will generate by iteration. While the complete solution of this problem is still to be accomplished, it is the purpose of this paper to develop methods of attack which will yield, in particular, an important necessary condition that each of two operations generate the other by iteration.

Definitions.

A bilinear operation is an operation $\varphi(x, y)$ of the form

$$\frac{A_\phi + B_\phi x + C_\phi y + D_\phi xy}{E_\phi + F_\phi x + G_\phi y + H_\phi xy}.$$

In this paper the coefficients will be supposed to be rational, ∞ will be considered a proper value and argument for a bilinear operation, and if $\varphi(x_1, y_1)$ is indeterminate as it stands, it shall be given the value $\lim_{x=x_1,\ y=y_1} \varphi(x, y)$ if this is determinate.

A bilinear operation will be said to be reduced if A_ϕ, $B_\phi + C_\phi$, D_ϕ, E_ϕ, $F_\phi + G_\phi$, and H_ϕ form a relatively prime set of integers.

Δ_ϕ is defined as

$$\left|\begin{vmatrix} A_\phi & B_\phi + C_\phi \\ E_\phi & F_\phi + G_\phi \end{vmatrix} \begin{vmatrix} A_\phi & D_\phi \\ E_\phi & H_\phi \end{vmatrix}\right| \left|\begin{vmatrix} A_\phi & D_\phi \\ E_\phi & H_\phi \end{vmatrix} \begin{vmatrix} B_\phi + C_\phi & D_\phi \\ F_\phi + G_\phi & H_\phi \end{vmatrix}\right|$$

The following definitions are made:

$$J_\phi(u, v) = A_\phi v^2 + (B_\phi + C_\phi)uv + D_\phi u^2,$$

$$K_\phi(u, v) = E_\phi v^2 + (F_\phi + G_\phi)uv + H_\phi u^2,$$

$$L_\phi = vJ_\phi - uK_\phi.$$

We shall term the roots of $L_\phi(u, 1) = 0$, $r_1(\varphi)$, $r_2(\varphi)$, $r_3(\varphi)$. We shall call a root of $L_\phi(u, 1) = 0$ free if it is not a simultaneous solution of $J_\phi(u, 1) = 0$ and of $K_\phi(u, 1) = 0$. If $\Delta_\phi \neq 0$, and $r_1(\varphi) \neq r_2(\varphi) \neq r_3(\varphi)$, $r_1(\varphi) \neq r_3(\varphi)$, φ will be said to be general, otherwise special. Clearly if φ is general r_1, r_2, and r_3 are all free.

An iterative field is a set of functions of n variables which is closed with respect to the following operations:

(1) permutation of any two arguments;

(2) substitution of two identical arguments for two different arguments;

(3) substitution of a function of the set for any argument.

Two operations are said to be iteratively equivalent if any iterative field containing either contains the other.

An iterative characteristic of an operation is a property which belongs to all operations iteratively equivalent to that operation.

Theorems.

I. *The free roots of* $L_\phi(u, 1) = 0$ *constitute iterative characteristics of* φ.

For $L_\phi(u, 1) = 0$ is equivalent to $\varphi(u, u) = u$, unless u is a simultaneous root of $J_\phi(u, 1) = 0$ and of $K_\phi(u, 1) = 0$. Now, $\varphi(u, u) = u$ is a property of φ invariant under iteration. If r is a root of $L_\phi(u, 1) = 0$ that is not free, then in general $\varphi(r, r) = r$ is not significant and determinate. If this latter equation is significant, let us subject our number system to a linear transformation that changes r to ∞. It may readily be seen that two operations will retain their iterative relationships unchanged under such transformations, that roots of $L_\phi(u, 1) = 0$ will remain roots, and that free roots will remain free. φ will then become a ψ of the form $A + Bx + Cy$. This operation can only be iteratively equivalent to operations of its own sort, for all of which ∞ is a root that is not free. This completes the proof of our theorem. As a corollary we obtain

II. *No general bilinear operation is equivalent to any special bilinear operation.*

III. *If* φ *and* ψ *are reduced equivalent general bilinear operations, then*

$$A_\phi = A_\psi,$$

$$B_\phi + C_\phi - E_\phi = B_\psi + C_\psi - E_\psi,$$

$$F_\phi + G_\phi - D_\phi = F_\psi + G_\psi - D_\psi,$$

$$H_\phi = H_\psi.$$

Lemma 1. *If* φ *is reduced,* u *and* v *can be chosen relatively prime in such a manner that* $J_\phi(u, v)$ *is prime to* $K_\phi(u, v)$.

Proof. Let N be the H.C.F. of A_ϕ, $B_\phi + C_\phi$, and D_ϕ. Let M be the product of all prime factors in Δ_ϕ but not in N. Let P be the product of all the primes in M, but not in A_ϕ nor in D_ϕ. Let Q be the product of all primes in M and A_ϕ, but not in D_ϕ. Let R be the product of all primes in M and D_ϕ, but not in A_ϕ. Let S be the product of all primes in N, but neither in E_ϕ nor in H_ϕ. Let T be the product of all primes in N and E_ϕ, but not in H_ϕ. Let U be the product of all primes in N and H_ϕ, but not in E_ϕ. Let $u = PRSU$, $v = QT$. We get as a result that J_ϕ is prime to M, while K_ϕ is prime to N. Now, it may readily be shown that any factor common to J_ϕ and K_ϕ belongs to Δ_ϕ. Since, however, any prime factor of Δ_ϕ belongs either to M or to N, it is either prime to J_ϕ or to K_ϕ.*

Lemma 2. *If a and b are any two numbers not both containing a prime p and if c and d, e and f, are couples of the same sort, while*

$$ad - bc \equiv 0 \pmod{p^k}, \qquad af - be \equiv 0 \pmod{p^k},$$

then

$$cf - de \equiv 0 \pmod{p^k}.$$

This lemma, whose proof is immediate, enables us to classify fractions into sets modulo p^k. We shall say that

$$a/b \equiv c/d \pmod{p^k}$$

if

$$ad - bc \equiv 0 \pmod{p^k}$$

while a and b do not contain p in common, nor do c and d.

Lemma 3. *If u/v and w/z are in their lowest terms, and*

$$u/v \equiv w/z \pmod{p^k},$$

then if φ is any bilinear operation such that

$$\varphi(u/v, u/v) \equiv u/v \pmod{p^k},$$

it follows that

$$\varphi(w/z, w/z) \equiv w/z \pmod{p^k}.$$

The proof of lemma 3 offers no difficulty.

Combining lemmas 2 and 3, it is at once clear that the equation

$$\varphi(u/v, u/v) \equiv u/v \pmod{p^k}$$

for a given u/v and p^k is an iterative characteristic of φ.

* Cf. Mathews, Theory of Numbers, vol. 1, p. 132.

We are now in a position to take up the proof of Theorem III. Clearly $\varphi(u/v, u/v)$ is equal to $J_\phi(u, v)/K_\phi(u, v)$. Let u and v be two numbers such that $J_\phi(u, v)$ and $K_\phi(u, v)$ are prime to one another. By lemma 1, such numbers exist. Then to say that

$$J_\phi(u, v)/K_\phi(u, v) \equiv u/v \pmod{p^k}$$

is equivalent to saying

$$L_\phi(u, v) \equiv 0 \pmod{p^k},$$

or that p^k is a factor of $L_\phi(u, v)$. If φ is iteratively equivalent to ψ, clearly we must have

$$L_\psi(u, v) \equiv 0 \pmod{p^k}.$$

That is, every prime power that divides L_ϕ must also divide L_ψ, or in other words, L_ϕ is a factor of L_ψ for a particular set of values of u and v. However, since by Theorem I the equations $L_\phi(u, 1) = 0$ and $L_\psi(u, 1) = 0$ have the same three distinct roots, L_ψ is a constant multiple of L_ϕ. Hence L_ϕ is always a factor of L_ψ. Likewise, L_ψ is always a factor of L_ϕ. Hence, apart from a possible difference of sign, L_ϕ and L_ψ are identical. This proves Theorem III.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
April 16, 1920.

NECESSARY AND SUFFICIENT CONDITIONS THAT A LINEAR TRANSFORMATION BE COMPLETELY CONTINUOUS.

BY PROFESSOR CHARLES ALBERT FISCHER.

(Read before the American Mathematical Society December 31, 1919.)

A LARGE part of the Fredholm theory of integral equations has been derived for the equation

$$h(x) = f(x) + \lambda T(f),$$

where $T(f)$ is a completely continuous linear transformation.* It has also been proved that every linear transformation, that

* F. Riesz, *Acta Mathematica*, vol. 41 (1918), pp. 71–98.

is, linear functional depending on a parameter, is equal to a Stieltjes integral of the form*

$$T(f) = \int_a^b f(y) d_y K(x, y). \tag{1}$$

In a former paper† I have found a necessary and some sufficient conditions that a linear transformation be completely continuous, and thus proved that this part of the Fredholm theory applies to Stieltjes integral equations of certain types. In the present paper will be found conditions which are both necessary and sufficient. The Volterra theory of Stieltjes integral equations has been discussed by Evans in his Cambridge Colloquium Lectures, 1916.

§ 1. *Preliminary Theorems.*

A set of functions is said to be compact if every infinite sequence of them contains an infinite subsequence which is uniformly convergent, and a transformation is completely continuous if it changes every bounded set of functions into a compact set.

The class $[f(y)]$ will be composed of all functions defined on the interval (a, b) which can be approached by monotone sequences, beginning with the class of all continuous functions. The definition of the Stieltjes integral has been extended by Young‡ to apply to such functions.

The function $K(x, y)$ will be assumed to satisfy the equation

$$K(x, y) = K(x, y + 0). \tag{2}$$

This will not affect the value of the integral (1), and when this equation is satisfied, $V_y K(x, y)$, that is, the variation of K considered as a function of y, is the least upper bound of $T(f)/\max|f|$.§ It follows that if $V_y K(x, y)$ is not bounded uniformly in x, the transformation (1) cannot be completely continuous, because in that case there would be a bounded set of f's for which the set $[T(f)]$ would not be bounded and therefore could not be compact.

* F. Riesz, *Annales Scientifiques de l'Ecole Normale*, ser. 3, vol. 31 (1914), pp. 9–14.

† This BULLETIN, vol. 25 (1919), p. 447.

‡ Young, *Proc. London Math. Society*, vol. 13 (1914), p. 109, or Daniell, *Annals of Mathematics*, vol. 19 (1918), p. 279.

§ Riesz, loc. cit., *Annales*; or Fischer, *Annals of Mathematics*, vol. 19 (1917), pp. 38–40.

It will now be proved that when $V_y K$ is bounded uniformly, every sequence of x's must contain a subsequence for which $K(x_n, y)$ approaches a uniquely determined function of y when n becomes infinite.

The positive constants $\epsilon_1, \epsilon_2, \cdots$ will be chosen so as to approach zero as a limit, and it will be assumed that $V_y K < M$. If there were a sequence of x's no subset of which would make the limit of $K(x_n, y)$ unique, there would be no subset which would make the difference between the largest and smallest limit points of $K(x_n, y)$ less than or equal to ϵ_1 uniformly in y, or else there would be such a subset which will be designated as $x_1^{(1)}, x_2^{(1)}, \cdots$. Then there would be no subset of the $x^{(1)}$'s which would make the same difference less than ϵ_2 uniformly, or else there would be such a subsequence, $x_1^{(2)}, x_2^{(2)}, \cdots$. It would not be possible to proceed in this way and get an infinite number of sequences, $x_1^{(i)}, x_2^{(i)}, \cdots$, each a subset of the proceeding, and such that the difference between the largest and smallest limit points of $K(x_n^{(i)}, y)$ would always be less than ϵ_i, because if this could be done the limit of $K(x_n^{(n)}, y)$ would be unique. Consequently there would have to be a subset of the original x's, $\bar{x}_1, \bar{x}_2, \cdots$, and a $k > 0$, such that no subsequence of the $\bar{x}$'s could make the difference between the largest and smallest limit points of $K(\bar{x}_n, y)$ less than or equal to k uniformly in y.

When a bounded set of functions is given there is always a subset of them that converges at any previously given denumerable set of points.* It follows that there would be a non-denumerable set of values of y for which the difference between limit points of $K(\bar{x}_n, y)$ would be greater than k, for a given sequence of the $\bar{x}$'s, since if there were only a denumerable set of such y's, the subset of the $\bar{x}$'s which made the above limit unique for these y's would make the difference between limit points of $K(\bar{x}_n, y)$ less than or equal to k for all values of y.

A subset of the $\bar{x}$'s could be taken such that $K(\bar{x}_n, y)$ would converge at a dense set of y's, and the other $\bar{x}$'s dropped. Then if y_1 were a point where two of its limit points differed by more than k, there would be two $\bar{x}$'s which would make K differ by more than k at y_1, while they made it differ by an arbitrarily small amount at a point arbitrarily near to y_1, and consequently $V_y K$ would be greater than $k/2$ in the neigh-

* Riesz, loc. cit., *Acta*, p. 93.

borhood of y_1 for at least one of these values of x. To state this more definitely, if a sequence $\delta_1, \delta_2, \cdots$ were given, approaching zero, there would be a sequence of $\bar{x}$'s which would satisfy the inequalities

$$V_y K(\bar{x}_n, y) \Big|_{y_1 - \delta_n}^{y_1 + \delta_n} > \frac{k}{2} \qquad (n = 1, 2, \cdots).$$

The other $\bar{x}$'s could then be dropped. Similarly there would be a y_2 and a subset of the remaining $\bar{x}$'s which would have the same property with respect to y_2, and the other $\bar{x}$'s could again be dropped. Proceeding in this way, the points $y_1, y_2, \cdots, y_N$ $(N > 2M/k)$ and a sequence of $\bar{x}$'s would be determined which would satisfy the inequalities

$$V_y K(\bar{x}_n, y) \Big|_{y_i - \delta_n}^{y_i + \delta_n} > \frac{k}{2} \qquad (i = 1, 2, \cdots, N).$$

If n were then taken so large that the intervals $(y_i - \delta_n, y_i + \delta_n)$ did not overlap, the inequalities

$$V_y K(\bar{x}_n, y) \geqq \sum_{i=1}^{N} V_y K(\bar{x}_n, y) \Big|_{y_i - \delta_n}^{y_i + \delta_n} > \frac{Nk}{2} > M,$$

would be satisfied. But this is contrary to the hypothesis that $V_y K < M$.

This completes the proof of the theorem: *If $V_y K$ is bounded uniformly in x, when a denumerable set of the x's is given, there must be a subsequence of them which makes $K(x_n, y)$ approach a unique limit as n becomes infinite.*

It will now be proved that if a set of functions converges, the variation of the limiting function cannot be greater than the limit of the variations of any sequence of the given functions.

If this were not the case there would be a sequence $\varphi_1(y)$, $\varphi_2(y), \cdots$, approaching a function $\varphi(y)$, and a $k > 0$, which would satisfy the inequalities

$$V\varphi(y) > V\varphi_n(y) + k \qquad (n = 1, 2, \cdots).$$

Then there would have to be a finite set of points $y_0 = a < y_1 < y_2 < \cdots < y_N = b$, which would satisfy the inequalities

$$(3) \quad \sum_{i=0}^{N-1} |\varphi(y_{i+1}) - \varphi(y_i)| > V\varphi_n(y) + \frac{k}{2} \qquad (n = 1, 2, \cdots).$$

But if n is large enough the inequalities

$$|\varphi_n(y_i) - \varphi(y_i)| < \frac{k}{4N} \qquad (i = 0, 1, 2, \cdots, N),$$

would be satisfied, and consequently inequality (3) would imply that

$$\sum_{i=0}^{N-1} |\varphi_n(y_{i+1}) - \varphi_n(y_i)| > V\varphi_n(y),$$

which is absurd.

§ 2. *Necessary and Sufficient Conditions.*

The following condition, which will be called condition A, will be proved to be necessary. Then a second condition, which will be called condition B, will be proved to be sufficient, and finally the two conditions will be proved to be equivalent.

CONDITION A. *A necessary and sufficient condition that the transformation* (1) *be completely continuous is that $V_y K(x, y)$ shall be bounded uniformly, and that when a sequence of x's is chosen in such a way that $K(x_n, y)$ converges, the equation*

$$\text{(4)} \qquad \lim_{n=\infty} V_y[K(x_n, y) - \lim_{n=\infty} K(x_n, y)] = 0$$

shall be satisfied.

If this condition is not satisfied there must be a sequence of x's for which $K(x_n, y)$ converges, and a $k > 0$, which satisfy the inequalities

$$\text{(5)} \qquad V_y[K(x_n, y) - \lim_{n=\infty} K(x_n, y)] > k.$$

If the functions $K(x_n, y) - \lim_{n=\infty} K(x_n, y)$ should also satisfy equations such as (2), there would be continuous functions $f_1(y), f_2(y), \cdots$, not greater than 1 in absolute value, which would satisfy the inequalities

$$\int_a^b f_m(y) d_y[K(x_m, y) - \lim_{n=\infty} K(x_n, y)] > k \qquad (m = 1, 2, \cdots).$$

Since this is not necessarily the case, it will be proved instead that when x_m is given there must be an $m' \geqq m$, and a con-

tinuous $f_m(y)$, such that $|f_m| \leqq 1$, and

$$\int_a^b f_m(y)d_y[K(x_{m'}, y) - \text{limit } K(x_n, y)] > \frac{k}{2}. \tag{6}$$

It follows from the last theorem in § 1, that when inequalities (5) are satisfied, when m is given there must be an $n > m$ which satisfies the inequality

$$V_y[K(x_m, y) - K(x_n, y)] > k.$$

Consequently there must be an $f_m(y)$ which satisfies the inequality

$$\int_a^b f_m(y)d_y[K(x_m, y) - K(x_n, y)] > k,$$

and at least one of the points x_m and x_n can be used for $x_{m'}$ in inequality (6).

The equation

$$\lim_{n=\infty} \int_a^b f(y)d_yK(x_n, y) = \int_a^b f(y)d_y \text{ limit } K(x_n, y)$$

must be satisfied if $f(y)$ is continuous.* Then when $f_m(y)$ is given there must be a finite N such that

$$\left| \int_a^b f_m(y)d_y[K(x_n, y) - \text{limit } K(x_n, y)] \right| < \frac{k}{6} \qquad (n \geqq N).$$

It will then be possible to select a sequence $\xi_1, \xi_2, \cdots$ from the given x's, and a sequence of continuous f's, which satisfy the inequalities, $|f_i| \leqq 1$, and

$$\left| \int_a^b f_i(y)d_y[K(\xi_j, y) - \text{limit } K(x_n, y)] \right| < \frac{k}{6}$$

$$(i = 1, 2, \cdots; j = i + 1, i + 2, \cdots),$$

and

$$\int_a^b f_j(y)d_y[K(\xi_j, y) - \text{limit } K(x_n, y)] > \frac{k}{2}$$

$$(j = 1, 2, \cdots).$$

* This follows from the proof of Bray's Theorem 3, *Annals of Mathematics*, ser. 2, vol. 20, pp. 180–181.

This implies that either

$$\left|\int_a^b [f_j(y) - f_i(y)]d_y K(\xi_j, y)\right| > \frac{1}{2}\left(\frac{k}{2} - \frac{k}{6}\right) = \frac{k}{6},$$

or else

$$\left|\int_a^b [f_j(y) - f_i(y)]d_y \text{ limit } K(x_n, y)\right| > \frac{k}{6}.$$

In either case there must be some value of x for which

$$|T(f_j) - T(f_i)| > \frac{k}{6}$$

$$(i = 1, 2, \cdots;\ j = i + 1, i + 2, \cdots),$$

and the transformation cannot be completely continuous. Therefore condition A is necessary.

CONDITION B. *A necessary and sufficient condition that the transformation* (1) *be completely continuous, is that when an* $\epsilon > 0$ *is chosen there must be a finite number of points* $\xi_1, \xi_2 \cdots, \xi_n$, *such that to every* x *in* (a, b) *there corresponds a* ξ_i *which satisfies the inequality*

$$V_y[K(x, y) - K(\xi_i, y)] < \epsilon. \tag{7}$$

If this condition is satisfied, and a decreasing sequence $\epsilon_1, \epsilon_2, \cdots$ is taken approaching zero, the ξ's corresponding to all the ϵ's can be arranged in one denumerable sequence. Every bounded set of f's must have a subset such that the functions

$$g_n(x) = T(f_n) = \int_a^b f_n(y)d_y K(x, y) \qquad (n = 1, 2, \cdots),$$

will converge at the denumerable set of points $\xi_1, \xi_2, \cdots$. This subset will now be proved uniformly convergent on the whole interval (a, b).

If an $\epsilon > 0$ is chosen there must be a finite N such that to every x on (a, b) there will correspond a ξ_i, $(i \leqq N)$, which satisfies inequality (7). Then if N' is so large that the inequalities

$$|g_n(\xi_i) - \lim_{n=\infty} g_n(\xi_i)| < \epsilon \qquad (i \leqq N;\ n \geqq N'),$$

are satisfied, the inequalities

$$|g_n(x) - g_n(\xi_i)| \leqq \max|f| \cdot V_y[K(x, y) - K(\xi_i, y)] \qquad (n = 1, 2, \cdots),$$

and inequality (7) will imply that

$$|g_n(x) - g_{n+m}(x)| \leqq 2[\max|f| + 1]\epsilon \qquad (n \geqq N';\ m = 1, 2, \cdots).$$

Consequently these g's must be uniformly convergent, and the transformation must be completely continuous.

If condition B is not satisfied there must be an $\epsilon > 0$ and an infinite sequence of x's such that

$$V_y[K(x_m, y) - K(x_n, y)] \geqq \epsilon \qquad (m \neq n).$$

It follows from the principal theorem of the first section, that if V_yK is bounded uniformly, a subset of these x's must make $K(x_n, y)$ convergent. As no subset of them can satisfy equation (4), condition A cannot be satisfied. Therefore when A is satisfied B must be, and since A is necessary and B sufficient, B must be necessary and A sufficient also.

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ON THE RELATION OF THE ROOTS AND POLES OF A RATIONAL FUNCTION TO THE ROOTS OF ITS DERIVATIVE.

BY MR. BEN-ZION LINFIELD.

(Read before the American Mathematical Society December 30, 1919.)

1. F. Lucas, *Journal de l'Ecôle polytechnique*, 1879, gave a mechanical proof of the following theorem:

The roots of the derivative of a cubic are the foci of the maximum ellipse inscribed in the triangle whose vertices are the roots of the cubic.

Professor Maxime Bôcher gave a simple proof of this theorem, *Annals of Mathematics*, volume 7, page 70, 1892. Here he made observations on the general theorem concerning the polynomial of nth degree and asked the question, "Could

not the first of these propositions be brought into connection with the focal properties of the higher plane curves?" Other demonstrations of Lucas's theorem have been given, but the writer is not aware of Professor Bôcher's question having been answered. It is the object of this note to answer this question by the following:

2. THEOREM. *The roots of the derivative of the rational function*

$$f(z) \equiv \prod_{i=1}^{n} (z - z_i)^{\mu_i}$$

are the multiple roots of $f(z)$ to one lower order and the foci of a curve of class $n-1$ which touches each segment z_iz_j $(i \neq j, = 1, 2, \cdots, n)$ at a point dividing it in the ratio of μ_i to μ_j.

3. Let

$$\alpha_i \equiv ux_i + vy_i + w \qquad (i = 1, 2, \cdots, n)$$

be the line equation of the point $P_i \equiv (x_i, y_i)$, where $z_i \equiv x_i + y_i\sqrt{-1}$ are the roots and poles of the rational function $f(z)$. Then,

$$\Pi \equiv \alpha_1\alpha_2\cdots\alpha_n = 0 \tag{1}$$

is the line equation of the n points x_i, y_i.

$$\varphi(u, v, w) \equiv \sum_{i=1}^{n} \mu_i \frac{\partial \Pi}{\partial \alpha_i} = 0 \tag{2}$$

is the tangential (line) equation of a curve of class $n-1$ tangent to the segments P_iP_j.

The point of contact of an arbitrary tangent to (2) whose coordinates are u, v, w, is given by equations

$$\frac{X}{\dfrac{\partial \varphi}{\partial u}} = \frac{Y}{\dfrac{\partial \varphi}{\partial v}} = \frac{1}{\dfrac{\partial \varphi}{\partial w}}.$$

Also from (2)

$$\frac{\partial \varphi}{\partial u} = \Sigma(\mu_j x_i + \mu_i x_j) \frac{\partial^2 \Pi}{\partial \alpha_i \partial \alpha_j},$$

$$\frac{\partial \varphi}{\partial v} = \Sigma(\mu_j y_i + \mu_i y_j) \frac{\partial^2 \Pi}{\partial \alpha_i \partial \alpha_j},$$

$$\frac{\partial \varphi}{\partial w} = \Sigma(\mu_i + \mu_j) \frac{\partial^2 \Pi}{\partial \alpha_i \partial \alpha_j}.$$

Therefore the coordinates of the point of contact of the tangent (u, v, w) with (2) are

$$X = \frac{\Sigma(\mu_j x_i + \mu_i x_j)\partial^2\Pi/\partial\alpha_i\partial\alpha_j}{\Sigma(\mu_i + \mu_j)\partial^2\Pi/\partial\alpha_i\partial\alpha_j},$$

$$Y = \frac{\Sigma(\mu_j y_i + \mu_i y_j)\partial^2\Pi/\partial\alpha_i\partial\alpha_j}{\Sigma(\mu_i + \mu_j)\partial^2\Pi/\partial\alpha_i\partial\alpha_j}.$$

Let the tangent (u, v, w) coincide with the tangent through the two points P_p and P_q. Then

$$\frac{\partial^2\Pi}{\partial\alpha_i\partial\alpha_j} = 0 \qquad (\text{when } i \neq p,\ j \neq q),$$

since $\alpha_p = 0$, $\alpha_q = 0$ and α_p or α_q occurs in each of these second derivatives except $\partial^2\Pi/\partial\alpha_p\partial\alpha_q$. Hence the coordinates of the point of contact of (2) with P_pP_q are

$$X = \frac{\mu_q x_p + \mu_p x_q}{\mu_q + \mu_p}, \qquad Y = \frac{\mu_q y_p + \mu_p y_q}{\mu_q + \mu_p},$$

dividing the segment P_pP_q in the ratio $\mu_p : \mu_q$.

4. The foci of the curve $\varphi(u, v, w) = 0$ are the roots of the polynomial $\varphi(-1, -\sqrt{-1}, z)$* where $z \equiv x + y\sqrt{-1}$. Thus in (2)

$$\alpha_i \equiv z - z_i, \qquad \Pi \equiv \prod_{i=1}^{n}(z - z_i).$$

Also

$$\varphi(-1, -\sqrt{-1}, z) \equiv \sum_{j=1}^{n}\mu_j \frac{\prod_{i=1}^{n}(z - z_i)}{z - z_i}.$$

But the derivative of $f(z)$ can be written

$$f'(z) = \prod_{i=1}^{n}(z - z_i)^{\mu_i - 1} \cdot \sum\mu_j \frac{\prod_{i=1}^{n}(z - z_i)}{z - z_j},$$

which establishes the theorem, § 2, as enunciated.

5. The present note is an abstract presenting the central theorem of a paper wherein the general curve $\varphi = 0$ of class

* Salmon, Higher Plane Curves, 3d ed., § 141; also Emch, this BULLETIN, vol. 25, pp. 157–161.

$n-1$ is discussed and methods evolved for drawing it. We go no further here than to notice the interesting case when $n = 3$, of which the theorem of Lucas is a particular case.

6. When $n = 3$ the roots of the derivative of

$$f(z) \equiv \prod_{i=1}^{3} (z - z_i)^{\mu_i}$$

are the roots of $\prod_{1}^{3} (z - z_i)^{\mu_i - 1}$ and the foci of

$$\varphi \equiv \mu_1\alpha_2\alpha_3 + \mu_2\alpha_1\alpha_3 + \mu_3\alpha_1\alpha_2 = 0.$$

The curve is clearly of the second class and therefore a conic. The center z_c is given by

$$\frac{\partial}{\partial z} \Sigma\mu_j \frac{\Pi(z - z_i)}{z - z_j} = 0,$$

or

$$z_c = \frac{(\mu_2 + \mu_3)z_1 + (\mu_3 + \mu_1)z_2 + (\mu_1 + \mu_2)z_3}{2(\mu_1 + \mu_2 + \mu_3)}$$

When the powers (μ) are positive the conic $\varphi = 0$ touches the sides of the triangle z_1, z_2, z_3 internally, and is therefore an ellipse inscribed in the triangle; the point of contact divides each side in the ratio of the powers of the vertices. If $\mu_1 = \mu_2 = \mu_3 = 1$, $f(z)$ becomes a cubic and the conic touches the sides of the triangle at their midpoints, which is Lucas's case.

When each power is not positive, it is only necessary to consider one as being negative, for changing the signs of all the powers does not effect $\varphi = 0$.

If $\mu_1 + \mu_2 + \mu_3 = 0$, the conic is a parabola, the center being at infinity. It follows from a well-known theorem, the focus of a parabola touching the three sides of a triangle lies on the circumcircle of the triangle, that in this case the roots of the derivative of $f(z)$ lie on a circle through the roots of $f(z)$.

A conic is clearly an ellipse or hyperbola according as it passes through a point between or not between two parallel tangents, respectively.

The tangents parallel to the sides of the triangle are easily found, since the center is known and the point of touch on a

side. Let the side z_1z_2 be the x-axis. The equation of the tangent parallel to this axis is

$$y = \frac{\mu_1 + \mu_2}{\mu_1 + \mu_2 + \mu_3} y_3,$$

with similar equations of the tangents parallel to the other two sides. Let μ_3 be negative. The points of contact with the sides z_1z_3 and z_2z_3 lie between the x-axis and the parallel tangent or outside them according as

$$\mu_1 + \mu_2 + \mu_3 \lesseqgtr 0,$$

and the conic is accordingly an ellipse or hyperbola.

Hence if all the powers be positive $\varphi = 0$ is an ellipse. If one of the powers be negative it is an ellipse, parabola or hyperbola according as their sum is less than, equal to or greater than zero. If two of the powers be negative it is an ellipse, parabola or hyperbola according as their sum is greater than, equal to or less than zero. In particular if $\mu_1 + \mu_3 = 0$ the side z_1z_3 is an asymptote. If $\mu_1 = \mu_2 = -\mu_3$ the sides z_1z_3 and z_2z_3 are asymptotes. If in addition the triangle $z_1z_2z_3$ is isosceles having its vertex at z_3, the circle having z_3 for its center and passing through z_1 and z_2 passes through the foci of $\varphi = 0$, for the altitude and half base of this triangle are the major and minor axes respectively.

UNIVERSITY OF VIRGINIA,
May 1, 1919.

MORITZ CANTOR, THE HISTORIAN OF MATHEMATICS.

BY PROFESSOR FLORIAN CAJORI.

(Read before the San Francisco Section of the American Mathematical Society June 17, 1920.)

PROFESSOR Moritz Cantor died at Heidelberg on April 10, 1920, in his ninety-first year, three days after the death of his only son. Considering his own long career and that of many other mathematicians, he could well remark on his eightieth birthday, "Die Mathematik gehört nicht gerade zu den ungesunden Handwerken."* His literary activity ex-

* Felix Müller, Der Mathematische Sternenhimmel des Jahres 1811, Leipzig und Berlin, 1911, p. 7.

tended over a very long period, from 1851 to 1908. He has been known for many years as the leading writer of his time on the general history of mathematics. In the nineteenth century, his Vorlesungen über Geschichte der Mathematik came to be regarded as the fullest and the most reliable general source of historical information on our science. Before his time the outstanding historian of mathematics had been the Frenchman, J. E. Montucla, whose history appeared in 1758 in two volumes, and in 1799–1802 in four volumes (completed with the cooperation of the astronomer J. J. Lalande). Moritz Cantor is the Montucla of the nineteenth century.

Cantor was born in Mannheim on August 23, 1829. In 1848 he attended the University of Heidelberg and later studied at Göttingen under K. F. Gauss and W. Weber.* In the fall of 1851 he took his doctorate at Heidelberg, presenting a thesis "Ueber ein wenig gebrauchtes Koordinatensystem," which as yet disclosed no interest in that field of research which later he cultivated so assiduously. Curtze adds: "Before he came to devote himself wholly to an academic career—a state's examination he never took—he went to Berlin for study, where he attended especially the lectures of Lejeune-Dirichlet." We have seen nothing to indicate that Cantor had been in contact with, or had been influenced by, the Heidelberg privat-docent and Lyceum professor, Arthur Arneth, the author of the Geschichte der reinen Mathematik, Stuttgart, 1852.† Of three or more articles which Cantor prepared in the next few years, only one short paper of a dozen pages is historical, viz., "Ueber die Einführung unserer Ziffern," 1856, published in the *Zeitschrift für Mathematik und Physik*. What was the circumstance that induced him to choose the history of mathematics as his life work? The present writer visited him on May 2, 1915, and, in the course of conversation, asked him this very question. He replied that he prepared a paper on Ramus, Stifel and Cardan,‡ which he read at a scientific meeting at

* For facts of his early life we depend upon an article by Maximilian Curtze in *Bibliotheca Mathematica*, ser. 3, vol. 1, Leipzig, 1900, pp. 227–231.

† See M. Cantor's sketch of Arneth in the Allgemeine Leutsche Biographie, Leipzig, 1875.

‡ The article is "Petrus Ramus, Michael Stifel, Hieronymus Cardanus, drei mathematische Charakterbilder aus dem 16. Jahrhundert," *Zeitschrift für Mathematik und Physik*, vol. 2, 1857, pp. 353–367.

Bonn and which was so well received that he felt encouraged to continue in historical work. Thus, a little praise proved in his case a deciding factor. Perhaps this experience led him later in life to deal leniently with the shortcomings of young writers and to show generous appreciation of any merit he was able to discover. Soon after, Cantor went to Paris and there met Michel Chasles and Joseph Bertrand. Chasles had published in 1837 his Aperçu Historique and enjoyed wide celebrity as the historian of geometry. He paid much attention to the young German and secured the publication in the *Comptes Rendus* of a historical note prepared by the latter. Cantor prided himself on the command that he acquired of the French language. He told the writer that after delivering an address in French at the Mathematical Congress of 1900 in Paris, two Frenchmen came up and asked him to decide a bet: One Frenchman arguing that no foreigner could learn to speak French so well; the other claiming that Cantor was a German living in Germany.

It was in 1863 that Cantor's first important historical book appeared, his Mathematische Beiträge zum Kulturleben der Völker, which attracted wide attention, and was both praised and criticized. It championed an oft-repeated but seldom realized ideal in the writing of the history of our science, namely, the exhibition of the place of mathematics in the cultural life of a people and in the intercourse between nations. In this book, and in the first, and part of the second, volumes of the Vorlesungen, Cantor kept this goal in view. In the discussion of arithmetic and the history of our numerals, as given in his Beiträge, Cantor was influenced not only by the French writers M. Chasles, A. J. H. Vincent, and Henri Martin, but also by the early papers of the German G. Friedlein. Later Cantor was in frequent correspondence with the great French historian, Paul Tannery.

In 1864 Maximilian Curtze of the gymnasium of Thorn sought the assistance of Cantor in the interpretation of passages in a medieval manuscript. Thus began a friendship which continued unbroken until the death of Curtze in 1903.

Having been privat-docent at Heidelberg since 1853, Cantor advanced in 1863 to the position of professor extraordinary of mathematics. Probably about this time he began to give lectures on the history of mathematics. These lecture

courses extended sometimes over a period of three semesters.* According to Curtze, among those attending Cantor's lectures was Siegmund Günther, who is well-known as a historian of science.†

For the next ten or twelve years Cantor's publications were mainly historical. We make special mention of his Euklid und sein Jahrhundert (1867) and his Die Römischen Agrimensoren (1875). These served as preliminary researches for his great work, the Vorlesungen über Geschichte der Mathematik, the first volume of which appeared in 1880 when he was over fifty years of age. His previous studies, extending over a period of a quarter of a century, had been preparatory for this standard work. In entering upon this gigantic task he showed a predilection far remote from that of Paul Tannery. Cantor was writing a general history, while Tannery preferred to exercise his genius in elucidating special fields and special periods. A second edition of this first volume appeared in 1894, a third in 1907. Twelve years elapsed between the first and second volumes of the Vorlesungen. This second volume (1892) carried the history from the year 1200 to the year 1668, when Leibniz and Newton were just entering upon their great careers of discovery. A second edition of the second volume appeared in 1899. During the years 1894–1898 were published successive parts of the third volume which carries the history down to 1758, the time of the rise of Lagrange in the mathematical firmament. A second edition was brought out in 1901. These three great volumes enjoyed an enthusiastic reception. In 1898 Professor George A. Gibson of Glasgow only moderately expressed the feeling of his time when he said:‡ "That the labour involved in collecting material and in reducing it to shape would be great, Mr. Cantor doubtless knew well; but in all probability his most liberal estimate of the demands likely to be made upon his energies has been far exceeded; in any case, one can readily understand the feelings of satisfaction with which he writes the preface to the concluding volume. It hardly requires to be stated that this history is certain to remain for many years the standard work on the

* *Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht* (Schotten), vol. 43, Leipzig und Berlin, 1912, p. 526.

† *Bibliotheca Mathematica*, ser. 3, vol. 1, p. 229.

‡ *Proceedings of the Edinburgh Mathematical Society*, vol. 17, 1898, p. 9.

subject with which it deals; in completeness, in accuracy, in clearness of arrangement, it stands unrivalled, and for the period which it covers is bound to be a permanent work of reference."

A great historian must be a judge and not an advocate. In this respect Cantor can be assigned a high place. In very few instances, if any, does national bias disclose itself. In this regard it would be difficult to name a historian of science who ranks above him.

At the time of the completion of the third volume, Cantor was sixty-nine years old. He knew that he was not equal to the task of preparing a fourth volume. On the other hand, the publishers and the mathematical public desired a continuation of the history. Finally, in 1904, at the Congress in Heidelberg, the plan was matured of bringing out a fourth volume on the cooperative plan. Nine men of different countries (V. Bobynin, A. v. Braunmühl, F. Cajori, S. Günther, V. Kommerell, G. Loria, E. Netto, G. Vivanti, C. R. Wallner) were selected to write certain parts under the direction of Cantor as editor-in-chief. Each collaborator was made responsible for his part. As far as possible each man was expected to follow the mode of presentation adopted in the previous volumes. The men were cautioned not to permit themselves to be dominated by preconceived ideas relating to the subjects or men treated in the history.* The fourth volume carried the history to 1799, the year of Gauss' doctor-dissertation.

It is the fate of a scientific man to be succeeded by others who carry his researches to a higher degree of perfection. Fraunhofer mapped the solar spectrum. His work was improved by E. Bequerel in France and J. W. Draper in America, both of whom used photography. Still greater perfection was introduced subsequently by A. Angström, H. Rowland, and others. Moritz Cantor had a similar experience. The twentieth century set new standards. High as his Vorlesungen towered above other general histories of mathematics (such as those of J. E. Montucla, C. Bossut (1802), A. G. Kästner (1796–1800), J. H. M. Poppe (1828), A. Arneth (1852), F. Hoefer (1874), M. Marie (1883–1888)), it came to be criticized with great severity for lack of accuracy

* For further details see *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 13 (1904), pp. 475–478.

in minute details. It became evident that the erection of a great and faultless architectural structure must be preceded by the painstaking labors of the quarryman. Gustav Eneström, the editor of the *Bibliotheca Mathematica*, devoted much of his time for fifteen years to recording corrections to the first three volumes of Cantor's Vorlesungen. The Herculean efforts of Cantor and the extremely penetrating criticisms of Eneström clearly point out two lessons to scholars of today: (1) The need of a more accurate general history of mathematics, prepared on the scale of that of Cantor's Vorlesungen and embracing the historical researches of the last twenty years; (2) the impossibility of this task for any one man. A history of the desired size and accuracy can be secured only by the cooperative effort of many specialists. The mode of preparation of Cantor's fourth volume points the direction to future success.

The prodigious literary activity of Moritz Cantor can be seen also on examination of the volumes of the *Zeitschrift für Mathematik und Physik* for the years 1856–1898, and the volumes of Allgemeine Deutsche Biographie. The mere enumeration of the titles of his book reviews and of his articles fills twenty-six pages.*

As previously stated, the present writer on May 2, 1915, visited Cantor at Heidelberg, who was then in his eighty-sixth year. Almost complete blindness had compelled him to stop research. He said he could see the general outline of a person's body, but could not make out the features. He could move about in his house without assistance. His hearing was still good and his mind fairly clear. Neither of us touched upon war issues, except that he once referred to me as coming from a *nominally* neutral country. Later in the day I met his son and daughter, who were free in expressing to me the hope that the United States would soon come to observe *real* neutrality. Cantor made inquiries about certain American and English mathematicians and spoke of special historical researches then in progress in Germany. It was at his suggestion, he said, that K. Bopp undertook the study of Gregory St. Vincent.

* See M. Curtze "Verzeichnis der mathematischen Werke, Abhandlungen und Recensionen des Hofrat Professor Dr. Moritz Cantor," in Abhandlungen zur Geschichte der Mathematik, Neuntes Heft, Leipzig, 1899, pp. 625–650.

He spoke of Siegmund Günther as one of his pupils and of the late A. v. Braunmühl as an indirect pupil. The former, he said, had offered himself for service in the war, but was too old to fight. Cantor spoke in high praise of Hermann Hankel, and declared that his early death had been a great loss to mathematics; would that he had lived long enough to have completed his history of mathematics! Hankel, he said, was excellent not only in subject matter but also in the style of presentation. He spoke of his own labors as those of a hewer of timber who with a big axe and with powerful strokes roughly cut the timber to proper form and dimension, but left it for those who follow him to dress, polish and finish. When mention was made of Zeno's arguments on motion, Cantor referred to Paul Tannery's brilliant discussion of that subject. He spoke of Caspar Wessel's noted paper on imaginaries as having been completely neglected for about a century, notwithstanding the fact that Wessel's name as well as the title of his paper were given in so prominent a publication as Poggendorff's Handwörterbuch; H. G. Zeuthen discovered Wessel by accident. Cantor was much surprised that the name Oughtred was pronounced in England "ootred," and said he could not recall for the moment another English word in which "ough" was pronounced "oo." He referred to the mathematical precocity of his young grandson who spoke of the cellar as "minus 1. Stock." He said that in 1900 he discontinued the Abhandlungen zur Geschichte der Mathematik as a department of the *Zeitschrift für Mathematik und Physik;* nevertheless, only recently, a Vienna scientist had sent him a book for review therein.

He remembered Simon Newcomb at a Congress in Paris delivering an address in the French language when all Germans would have preferred to hear him speak English. At the Congress in Rome an Englishman arose, he said, to thank the King and the Italian mathematicians for the royal reception accorded to members of the Congress. The one who made reply for the Italian government declared his great admiration for the language of William Shakespeare, yet had to confess that his insufficient command of that language prevented him from following the Englishman's speech. A friend nudged the Italian speaker and called his attention to the fact that the Englishman's speech had been delivered in the Italian language. In speaking of Georg Cantor, who

had shortly before celebrated his seventieth birthday, he said that Georg Cantor's ancestors, as well as his own, came originally from Denmark, that, although he thought he and Georg Cantor were distantly related, the latter did not agree with him. After a dinner which was served in the garden in the open, the writer, having been told that it was Cantor's habit to lie down after the mid-day meal, departed.

UNIVERSITY OF CALIFORNIA.

SHORTER NOTICES.

Les Spectres numériques. By MICHEL PETROVITCH, with a preface by EMILE BOREL. Paris, Gauthier-Villars, 1919.

THE "spectral method" of Professor Petrovitch will be understood through an example taken from the last chapter of his monograph.

Let it be required to determine the coefficients in the development of $(1 + z)^6$. We have

$$(1.01)^6 = 1.061520150601,$$

and the binomial coefficients sought appear in the decimal above, some of the coefficients being separated by zeros.

It is clear that a similar method can be used to determine any finite set of positive integers, if the integers are the coefficients in the development of a known rational integral function, and that generalizations to infinite sequences of integers are possible.

Professor Petrovitch calls the decimal above a "spectrum" of the function $(1 + z)^6$, comparing the binomial coefficients to the colored portions of the spectrum, and the zeros to the dark portions. The book, up to the last chapter, is given up to a discussion of methods for associating "spectra" with different classes of functions.

In spite of the amiable preface of Borel, nothing will be found in the work to make one feel that the author has a point of view which is likely to lead to fruitful developments. Professor Petrovitch has certain speculations, but they are not of enough merit to warrant this address to the mathematical public.

Throughout the book are scattered references to results of contemporary analysts, and these contribute a certain amount of animation to the text, although the theorems quoted lead only to trivial conclusions in regard to the matter in hand. The style of the book is sufficiently elegant, and would be worthy of a more valuable scientific production.

J. F. RITT.

La Part des Croyants dans les Progrès de la Science au XIX^e Siècle. By ANTONIN EYMIEU. Première partie: Dans les sciences exactes. Troisième édition. Paris, Perrin et Cie., 1920. ii + 272 pp. Price 5 francs.

THIS little volume of entertaining but not particularly scientific summer reading is of interest to the mathematician chiefly because of the biographical material contained in the first chapter. It had its inception in connection with an anecdote which is often told of a verbal encounter in the Chamber of Deputies some thirty years ago. M. Charles Dupuy, speaking as rapporteur de l'Instruction publique, had spoken of the fatal anæmia of the facultés catholiques which, when they entered upon scientific studies, reached at last a stage where their faith called out, "Tu n'iras pas plus loin." The remark, so the official journal of the day records, was hailed with cries of approval from the Right, at which manifestation an opponent exclaimed, "As if there had never been any Christian scholars!" To this M. Dupuy replied, amid laughter, that it would be an interesting thing to see the list.

What M. Eymieu proposed for himself some five years ago, when the first edition appeared, was to meet the challenge and to show that it was those of religious faith who made the greatest contributions to the exact sciences in the nineteenth century. He admits, however, that it is an impossibility to prepare a complete catalogue of scientists and of their religious beliefs, and so he sets about to furnish a brief list, limited to the greatest contributors to mathematics, astronomy physics, and chemistry.

In mathematics M. Eymieu has selected the names of Gauss, Cauchy, Poincaré, Lagrange, Abel, Galois, Riemann, Weierstrass, and Hermite as representing the great research scholars—"les grand initiateurs," of whom, as he says, "de l'aveu de tous les bons juges, trois . . . dominent son

histoire," namely, Gauss, Cauchy, and Poincaré. It is of interest to observe that two out of these three who are said to have dominated the mathematical history of the century are French. Of the nine great leaders, four are also of the author's nationality, or five if we count Lagrange, who spent the latter part of his life in Paris and who was of French extraction; three are German, and one is a Norwegian. Out of the nine, the author proves to his satisfaction that five and probably six were religious men, that two (Lagrange and Galois) had no interest in religion, and that there remains a doubt as to the position of Poincaré. The two leading mathematical astronomers he takes to have been French, namely, Laplace and Leverrier, and each he claims to have been possessed of undoubted religious faith.

What is chiefly interesting in the work is that which the French call the "causerie" in which the author indulges. It is his gossip, his anecdotes about men, and his happy method of quoting interesting passages from speakers like Poincaré, Arago, and Bertrand that make the book readable. Such a human touch is seen in his quotation from that troublesome young Bolshevik, Galois, who was too advanced even for a France that claimed to stand for the most progressive thought of a century ago. The incident is well known, but is always interesting when one considers the sudden closing up of a life that was mentally brilliant and physically and morally a failure. The night before the duel in which he met his death he wrote to his republican patriots: "Je meurs victime d'une infâme coquette et de deux dupes de cette coquette. C'est dans un misérable cancan que s'eteint ma vie. . . . Adieu, j'avais bien de la vie pour le bien public."

M. Eymieu has been more successful in his selection of the nine great leaders, however open to criticism this may be, than in his choice of other great mathematicians who were distinctly religious in their faith. It is with some surprise that, among these "grands mathématiciens qui furent aussi de grands croyants," one finds that Boncompagni's name "leads all the rest." Boncompagni was a great man, and a great historian of mathematics, but he was not a great mathematician. The second name in the minor list is that of Joseph Bayma "que les journaux américains, en annonçant sa mort, proclamaient 'le géant des mathématiques,'"—after which statement it is quite unnecessary, at least for purposes of review, to examine the list any further.

The fact is that M. Eymieu has done his cause no good in the eyes of a scientific reader. His selections of men have not, in general, been made with care; indeed, they have not been made with ordinary knowledge. He has not scientifically gone to work to secure his information, as witness his uncertain results concerning the faith of Simon Newcomb. He has simply set about to support the belief of the uneducated or the half educated man of his own religious faith. It cannot be expected that he should have done for the dead what Professor Leuba did with respect to the religious beliefs of living scientists, but no one who has worked in the history of mathematics can fail to see that a much stronger case could have been made, and legitimately made, if the author had studied the problem with greater care.

It is evident to everyone that the most difficult thing to weigh in a scientific balance is the religious belief of mankind. The reasons are equally evident. One thing is clear, however,—that the study of the exact sciences no more tends to lessen this religious faith than the study of commerce, of civics, of sociology, or even of theology. The history of the exact sciences offers abundant illustrations of this fact, and evidence of a more convincing kind than that which M. Eymieu has adduced. Indeed, it would be a strange and inexplicable thing if scientific investigation should fail to show that mathematics, that branch of knowledge which is continually in touch with the infinite and is continually revealing the mysteries of the eternal, should fail to foster religious faith to an extent not reached by the other subjects of human study.

DAVID EUGENE SMITH.

The Early Mathematical Manuscripts of Leibniz. By J. M. CHILD. Chicago, 1920. iv + 238 pp.

THIS important work consists of translations of various Latin manuscripts of Leibniz found by Dr. C. I. Gerhardt in the Royal Library of Hanover about seventy-five years ago. These manuscripts were published by Dr. Gerhardt as parts of three works which he wrote on the origin of the differential and integral calculus,* and have long been known in their

* Historia et Origo Calculi Differentialis, a G. G. Leibnizio conscripta, Hanover, 1846.

Die Entdeckung der Differentialrechnung durch Leibniz, Halle, 1848.

Die Geschichte der höheren Analysis; erste Abtheilung: Die Entdeckung der höheren Analysis, Halle, 1855.

Latin form to scholars who cared to consult these sources. There is no denying the fact, however, that the present generation of students in this country has not been trained to look upon Latin as a medium of intellectual exchange, and even in England it is undoubtedly true that the language has lost enough of its former prestige to make a translation of such material as the early manuscripts of Leibniz a great convenience to anyone who is interested in the subjects treated. For this reason there can be no question as to the value of this work and as to the renewed interest which it will awaken in the problem of the actual contribution of Leibniz to the revealing of the laws of the calculus and to the invention of a convenient symbolism.

The first document translated is the Historia et Origo, which Gerhardt published in 1846. This article is written in the third person, in the style which American readers have recently seen in the autobiography of Henry Adams, and was probably intended for anonymous publication. The story is here told, in popular fashion, of the steps which led Leibniz to his discovery. His disturbance over the publication of the Commercium Epistolicum in 1712, a publication which everyone who has examined it must admit is not a judicial document, seemed to him to require an answer. For this reason he proceeded to relate the incidents of his early publication, at the age of twenty, of the De Arte Combinatoria (1666); of his interest at the same time in general questions of analysis, including the theory of finite differences; of his visit to London at the age of twenty-six; of his acquaintance with Huyghens; and of the gradual development of his ideas of the calculus. He summarizes his case against the British school in the following words:

"Since therefore his opponents, neither from the Commercium Epistolicum that they have published, nor from any other source, brought forward the slightest bit of evidence whereby it might be established that his rival used the differential calculus before it was published by our friend;* therefore all accusations that were brought against him by these persons may be treated with contempt as beside the question."

The second body of translated material consists of several manuscripts of the period 1673–1680. These relate to various topics bearing upon the calculus. For example, on November

* That is, by Leibniz himself.

1, 1675. he treats of moments about axes, giving none of the new symbols, *omn.* being still used instead of $\int$. Ten days later (November 11), the symbol $\int$ is used, x/d and dx are stated to be interchangeable, differentials of the second order are rejected, the problem is solved of finding a curve such that the rectangle contained by the subnormal and ordinate is constant, and Barrow's form of the differential triangle is used. Ten days later still, November 21, he speaks of a new kind of trigonometry of indivisibles, in which reference the editor finds the influence of both Pascal and Barrow. In June, 1676, he writes that "the true general method of tangents is by means of differences." In the following month (July, 1676) he writes, partly in Latin and partly in French, on the inverse method of tangents, and in November of the same year he sets forth a number of the basic laws for differentiation and integration. In July, 1677, however, he shows no evidence of ability to differentiate logarithmic, exponential, or trigonometric functions, but by 1680 he seems to have been in possession of the general theory which he published in the *Acta Eruditorum* in 1684.

This brief summary will serve to give some idea of the material awaiting the further study of scholars;—further study, because the world will not be inclined to accept as satisfactory the study given by either Dr. Gerhardt or Mr. Child. Dr. Gerhardt was a careful student, but he is shown by Mr. Child not to have been altogether judicial in his statements. As to the editor of the present work, two questions will occur to any reader who carefully examines his contribution. In the first place, has he approached the subject with the unprejudiced mind of a searcher after truth? Mr. Child has himself answered this question (page 229):

"I therefore set out with the determination to break down, if possible, the credibility of Leibniz as a witness in his own defense, when it came to unimportant details; then to show that he had opportunities for obtaining everything necessary to the development of the calculus, that he could not be expected to supply for himself by original work, without having need to know anything of the work of Newton; then to show that these sources of information were set out in a form far more suitable to the requirements of Leibniz than the work of Newton; finally, to clinch the matter, that the analogy of Leibniz's work was so close to these sources, that it was idle to suppose that he made use of any other sources."

It is impossible to read this confession of prejudice without being conscious of a slight feeling of, or akin to, that of amusement in Mr. Child's strictures upon Gerhardt:

"Never surely did any man have such a glorious opportunity as Gerhardt, in the whole history of scientific controversies; surely there never was an advocate who left himself so open to the attacks of the opponents."

It is to be regretted that Mr. Child repeats so often his ideas of the dependence of Leibniz upon Barrow. This repetition is partly due no doubt to the fact that the book is made up of essays that appeared from time to time in *The Monist*, little effort having been made to unify the presentation of the matter when combined in book form. It is also to be regretted that a more restrained style was not possible, since such a style would have carried much greater conviction than the one adopted. Probably the best approach to a perfect description of the working of a human mind in the reaching of a mathematical discovery is that given by Lord Moulton in his address at the Napier celebration at Edinburgh in 1914. There the trained intellect of a mathematician and an eminent jurist concentrated on giving a clear analysis of the development of a great idea, and the result of the analysis was a masterpiece,—delightful in style, free from any apparent bias, and convincing in its conclusions.

It is too much to expect, however, that we can all be Lord Moultons. Perhaps it is more human to find ourselves influenced by Carlyle as we see him in his thoroughly biased essay on Cromwell. But the reader will soon find that with all of Mr. Child's evident scholarship and painstaking research, and in spite of all our indebtedness to him for his excellent translation and the information which he often gives us, in the matter of style he is not particularly fortunate. Illustrations like the following will tend to convince the reader, however generous may be his inclinations, that the work was not carefully revised before publication:

"Does not this silence on the part of Tschirnhaus, the personal friend of Leibniz, rather tend to make Leibniz's plea, that his opponents had had the shrewdness to wait till Tschirnhaus, among others, was dead, recoil on his own head, in that he has done the very same thing?" (Page 29.)

"The work of Descartes, looked at at about the same time as Clavius, that is, while he was still a youth, 'seemed to be more intricate.'" (Page 37.)

"This, without either proof or figure, is a hopeless muddle. . . . Goodness knows what the use of it was supposed to be in this form!" (Page 61.)

"Neither Gerhardt nor Weissenborn tried to get to the bottom of these manuscripts, being content with simply 'skimming the cream.'" (Page 74.)

"Thus what is generally considered to be a muddle turns out to be quite correct. The muddle is not with Leibniz, it is with the transcriber." (Page 81.)

"This is of course nonsense." (Page 97.)

"I cannot get out of my head the suggestion that . . . ." (Page 110.)

"Is Leibniz trying to draw a red herring across the trail, the real trail that leads to Barrow's a and e?" (Page 128.)

Unfortunately, there are a large number of similar instances that will strike the reader's attention as he studies the pages of the book.

It is a rather low type of criticism that looks only for the misprints and inconsequential slips of the pen in the work of an author. When Mr. Childs remarks that "there is of course the usual misprint (in Gerhardt's work) that one is becoming accustomed to," he tempts the reader, however, to recall various instances of a similar kind in the work under review. Without wishing to call attention to these misprints in detail, the point may be illustrated by the cases of 15 for 16 (page 31), the period for a comma on page 74 (line 5), the absence of an interrogation point after the question in the note on page 101, and the date 1874 for 1674 as that of the second edition of Barrow (page 13).

That the book is a valuable contribution to the history of mathematics, however, is evident to anyone who gives its pages even a casual reading.

DAVID EUGENE SMITH.

An Enquiry Concerning the Principles of Natural Knowledge. By A. N. WHITEHEAD. Cambridge University Press, 1919. xii + 200 pp.

THE aim of this work is to illustrate the principles of natural knowledge by an examination of the data and the experiential laws fundamental for physical science. The modern theory of relativity has opened the possibility of a new answer to the question as to how space is rooted in experience and has

brought to light a new world of thought as to the relations of space and time to the ultimate data of perceptual knowledge. "The present work is largely concerned with providing a physical basis for the more modern views which have thus emerged. The whole investigation is based on the principle that the scientific concepts of space and time are the first outcome of the simplest generalizations from experience, and that they are not to be looked for at the tail end of a welter of differential equations."

Three main streams of thought—the scientific, the mathematical, and the philosophical—are relevant to the theme of the enquiry. About half the book is given to parts I and II on the traditions of science and the data of science respectively. In part III on the method of extensive abstraction we have a philosophical and postulational treatment of the space-time manifold; and this is employed in part IV to yield a theory of objects. The fundamental assumption elaborated "is that the ultimate facts of nature, in terms of which all physical and biological explanation must be expressed, are events connected by their spatio-temporal relations, and that these relations are in the main reducible to the property of events that they contain (or extend over) other events which are parts of them."

R. D. CARMICHAEL.

Differential Equations. By H. BATEMAN. Longmans, Green and Company, London and New York, 1918. xii + 306 pp.

"THE subject of Differential Equations has grown so rapidly in recent years that it is difficult to do justice to all branches of the subject in a single volume." So reads the opening sentence of the preface to the book under review. The statement is literally true; but it is nevertheless of such form that its connotation may be misleading to the learner. One who is not acquainted with the subject of differential equations may justly conclude from this sentence that it is possible, though indeed difficult, to do justice to all branches of the subject in a single volume. And yet it is probable that no one who knows the field would be willing to maintain such a judgment. It is in fact true that it would be difficult to do justice to all branches of the subject in ten volumes of the size of the one under consideration.

In the second sentence of his preface the author indicates

the purpose of his book in the following language: "In writing this book I have endeavored to supply some elementary material suitable for the needs of students who are studying the subject for the first time, and also some more advanced work which may be useful to men who are interested more in physical mathematics than in the developments of differential geometry and the theory of functions."

From this it will be seen that the author in his single exposition seeks to serve two rather different purposes. In so far as these purposes diverge there is an evident possibility that the provision for one will interfere with the best provision for the other. It seems to the reviewer, in fact, that the more advanced matter has been provided in such way as to interfere with the usefulness of the book to students who are studying the subject for the first time. There does not appear to be a sufficient indication (even with the help of the preface) as to what parts of the book are best suited to the needs of the beginner. It is not clearly apparent, indeed, that the author himself had in mind a definite separation of material for serving the two purposes named. Moreover, the exposition from time to time assumes on the part of the reader a range of mathematical knowledge which is not the common possession of students pursuing the subject for the first time. A similar statement may also be made relative to those who are interested primarily in physical mathematics.

These considerations may lead to the fear that the usefulness of the book has not been increased by the attempt to serve simultaneously two distinct purposes which are not so intimately bound together as to justify this plan of treatment without a more distinct separation and indication of parts to serve these purposes.

Occasionally the rhetorical relations within a sentence might be improved. Greater care in proofreading would have resulted in the avoidance of several minor blemishes. But so far as the reviewer has observed these defects are not likely to cause the reader serious inconvenience.

Having said these things, let us turn now to matters which it is more pleasant to dwell upon. Even a rapid examination of the contents of the volume will bring out the fact that it contains a considerable amount of useful material not easily accessible elsewhere, and indeed some new material in Chapters VII and VIII.

Chapter II (pages 17–59) on integrating factors contains a rather wide range of useful and interesting matter, a part of which one would hardly expect to find under the title given. Further elementary methods of solution are treated in Chapter III (pages 60–86) under the heading transformations. In Chapter V (pages 103–115) we have a brief treatment of differential equations with particular solutions of a specified type. Some rather interesting results are obtained by starting from this elementary point of view, results which it would probably be more difficult to secure by any other method. They have to do especially with the algebraic integrals of certain important equations. Chapters VI to XI bear the following titles: partial differential equations, total differential equations, partial differential equations of the second order, integration in series, the solution of linear differential equations by means of definite integrals, the mechanical integration of differential equations.

Chapter VIII (pages 169–222) on partial differential equaions of the second order will be found particularly rich in material with an appeal to those who are interested primarily in physical mathematics. Here one finds a considerable treatment of the equation of wave propagation in three dimensions, of the Maxwell equations, of the electron equations, of Laplace's equation, and of the equation of the conduction of heat.

The chapter on solutions by means of definite integrals contains a useful summary of material, a part of which is not so conveniently found elsewhere so far as the reviewer is aware.

Those who give instruction to elementary classes in differential equations will find in this book of Bateman's a useful source of supplementary material of the nature both of additional topics for special report by members of the class and of problems to furnish a variation from those in the basic text. Some of the problems here included are rather unusual in character (as for instance 2 and 6 and 17 of the miscellaneous examples) and hence are of value to the instructor who wishes to introduce greater variety.

R. D. CARMICHAEL.

NOTES.

THE concluding (June) number of the *Annals of Mathematics* contains the following papers: "Algebraic surfaces, their cycles and integrals, by S. LEFSCHETZ; "The potential of ring-shaped disks," by E. P. ADAMS; "Total differentiability. A correction," by E. J. TOWNSEND; "Existence theorem for the non-self-adjoint linear system of the second order," by H. J. ETTLINGER; "Motion in a resisting medium," by J. K. WHITTEMORE; "Continuous matrices, algebraic correspondences, and closure," by A. A. BENNETT; "Urn schemata as a basis for the development of correlation theory," by H. L. RIETZ; "On pseudo-resolvents of linear integral equations in general analysis," by T. H. HILDEBRANDT.

THE following advanced courses in mathematics are offered at the Italian universities during the academic year 1920–1921:

UNIVERSITY OF BOLOGNA.—By Professor P. BURGATTI: Principles of celestial mechanics; figures of equilibrium of rotating fluid masses; problems of cosmogony, three hours.—By Professor L. DONATI: Elementary and general relativity, three hours.—By Professor F. ENRIQUES: Elementary mathematics in the light of higher concepts. History of the ideas; criticism; problems, three hours.—By Professor S. PINCHERLE: Analytic functions; integral analytic functions; ordinary differential equations from Lie's standpoint, three hours.

UNIVERSITY OF CATANIA.—By Professor M. CIPOLLA: Theory of groups of finite order with applications, four hours.—By Professor G. SCORZA: Enumerative geometry; theory of invariants, five hours.—By Professor ——: Mathematical physics, three hours.

UNIVERSITY OF GENOA.—By Professor G. LORIA: Infinitesimal geometry of common space and of hyperspaces, three hours.—By Professor C. SEVERINI: Theory of analytic functions, four hours.—By Professor O. TEDONE: Elementary theories of electricity and magnetism, three hours.

UNIVERSITY OF MESSINA.—By Professor P. CALAPSO: Functions of a complex variable and elliptic functions, four

hours.—By Professor G. GIAMBELLI: Singularities of algebraic curves; introduction to the geometry on an algebraic curve, four hours.—By Professor O. LAZZARINO: Vector operations and functions; fields of force; principles of electrostatics and of the statics of elastic bodies, four hours.

UNIVERSITY OF NAPLES.—By Professor F. AMODEO: History of mathematics: the century of Galileo Galilei, three hours.—By Professor A. DEL RE: Analytic theory of heat, four and one-half hours.—By Professor R. MARCOLONGO: Electrodynamics; theory of Lorentz; optical applications, three hours.—By Professor D. MONTESANO: Geometry of straight lines and of conics in the ordinary spaces, six hours.—By Professor E. PASCAL: Abelian integrals and functions, three hours.

UNIVERSITY OF PADUA.—By Professor U. AMALDI: Contact transformations and canonical systems, four hours.—By Professor F. D'ARCAIS: Monogenic functions; gamma and elliptic functions, four hours.—By Professor P. GAZZANIGA: Theory of numbers, three hours.—By Professor G. RICCI: Methods of absolute differential calculus, with applications to the theory of elasticity, four hours.—By Professor F. SEVERI: Calculus of probabilities, four hours.—By Professor A. TONOLO: Partial differential equations of the first order, three hours.

UNIVERSITY OF PALERMO.—By Professor G. BAGNERA: Complex variables and integral analytic functions; elliptic functions, three hours.—By Professor M. DE FRANCHIS: A study of the real parts of algebraic entities, three hours.—By Professor M. GEBBIA: Vector fields, electro- and magnetostatics, four and one-half hours.—By Professor A. SIGNORINI: Theory of relativity, three hours.—By Professor V. STRAZZERI: Riemann surfaces; algebraic curves; abelian integrals, three hours.

UNIVERSITY OF PAVIA.—By Professor L. BERZOLARI: Abelian integrals, three hours.—By Professor U. CISOTTI: Relativity; Einstein's mechanics, three hours.—By Professor F. GERBALDI: Functions of a complex variable and elliptic functions, three hours.—By Professor F. SIBIRANI: Plane and twisted curves; principles of the theory of surfaces and con-

gruences of rays, three hours.—By Professor G. VIVANTI: Theory of functions of a real variable; Lebesgue integrals, three hours.

UNIVERSITY OF PISA.—By Professor G. ARMELLINI: Celestial mechanics, three hours.—By Professor E. BERTINI: Projective geometry of hyperspaces, three hours.—By Professor L. BIANCHI: Functions of a complex variable; algebraic numbers and analytical arithmetic, three hours.—By Professor G. A. MAGGI: Analytical dynamics; equilibrium and vibration of elastic bodies; elastic theory of light, four and one-half hours.

UNIVERSITY OF ROME.—By Professor G. BISCONCINI: Geometric applications of the calculus, three hours.—By Professor E. BOMPIANI: Continuous groups of transformations, three hours.—By Professor F. CANTELLI: Mathematical statistics, three hours; Actuarial mathematics, three hours.—By Professor G. CASTELNUOVO: Complex variables and algebraic functions, three hours.—By Professor U. CRUDELI: Introduction to advanced theories of electricity and magnetism, three hours.—By Professor T. LEVI-CIVITA: Absolute differential calculus with applications, three hours.—By Professor A. PERNA: Complements of mathematical analysis, three hours.—By Professor L. SILLA: Differential equations of dynamics, three hours.—By Professor V. VOLTERRA: General relativity, three hours; Differential equations of mathematical physics, three hours.

UNIVERSITY OF TURIN.—By Professor T. BOGGIO: Hydrodynamics, three hours.—By Professor G. FUBINI: Differential geometry and continuous groups, with special reference to the groups of motions and of conformal and projective transformations.—By Professor C. SEGRE: Geometry of differential equations, three hours.—By Professor C. SOMIGLIANA: Theory of oscillations and electromagnetic optics, three hours.—By Professor G. TOGLIATTI: Projective differential geometry, three hours.

The following courses are announced in American universities for the academic year 1920–1921, in addition to those listed in previous numbers of the BULLETIN.

UNIVERSITY OF ILLINOIS.—By Professor E. J. TOWNSEND: Complex variables, three hours; Differential equations, three hours.—By Professor G. A. MILLER: Theory of groups (introductory course), three hours; Theory of equations, three hours (first term).—By Professor J. B. SHAW: Vector methods, three hours.—By Professor A. B. COBLE; Algebraic and abelian functions, three hours.—By Professor R. D. CARMICHAEL: Linear difference equations, three hours.—By Professor A. EMCH: Geometric transformations, three hours (first term); Geometry in a complex field, three hours (second term).—By Professor A. R. CRATHORNE: Theory of statistics, three hours.—By Professor G. E. WAHLIN: Theory of numbers, three hours.—By Professor H. BLUMBERG: Theory of aggregates, three hours (second term).—By Professor E. B. LYTLE: History of mathematics, two hours (second term).

YALE UNIVERSITY.—By Professor E. W. BROWN: Differential equations, three hours (second term); Advanced mechanics, three hours.—By Professor J. PIERPONT: Theory of functions of real variables, two hours; Theory of functions of a complex variable, three hours.—By Professor W. R. LONGLEY: Theory of differential equations, two hours.—By Professor E. J. MILES: Advanced calculus, three hours (first term); Differential geometry, two hours.—By Professor J. I. TRACEY: Modern and differential geometry, three hours; Analytic geometry, two hours.—By Professor J. K. WHITTEMORE: Advanced differential geometry, two hours.—By Dr. W. L. CRUM: Mechanics, three hours.—By Mr. J. S. MIKESH: History of mathematics, two hours.—By Dr. J. M. STETSON: Higher algebra, two hours.

THE class of sciences of the Belgian academy announces the following subjects for prize memoirs for 1921: (1) a contribution to the study of the properties of analytic functions that do not take certain values in a given domain; (2) a contribution to the study of birational transformation in a space of more than two dimensions.

THE adjudicators of the Hopkins prize of the Cambridge philosophical society have made the following awards: for the period 1903–06, to Dr. W. BURNSIDE, of Pembroke College,

for investigations in mathematical science; for the period 1906–09, to Professor G. H. BRYAN, of Peterhouse, for investigations in mathematical physics, including aerodynamic stability; and for the period 1909–12, to Mr. T. R. WILSON, of Sidney Sussex College, for investigations in physics, including the paths of radio-active particles.

PROFESSOR H. HAPPEL, of the University of Tübingen, has been appointed to a professorship at the Breslau technical school.

AT the University of Münster, Professor R. COURANT, of the University of Göttingen, has been appointed professor of mathematics. Associate professor H. KONEN has been promoted to a full professorship of theoretical physics.

AT the University of Kiel, Dr. O. TOEPLITZ has been promoted to a full professorship of mathematics, and Dr. E. MADELUNG to a full professorship of theoretical physics. Professor H. JUNG has resigned, to accept an appointment as professor of mathematics at the University of Halle.

THE Braunschweig technical school has conferred the honorary degree of doctor of engineering on Professor H. LORENZ, of the Danzig technical school.

AT the University of Rostock, associate professor R. WEBER, of the department of applied mathematics and physics, has been promoted to an honorary professorship; Dr. O. HAUPT, of the Karlsruhe technical school, has been appointed at a professorship; Professor O. STAUDE has received the honorary degree of doctor of engineering from the technical school of Darmstadt.

ASSOCIATE professor R. WEITZENBÖCK, of the German technical school at Prague, has been promoted to a full professorship of mathematics.

THE following persons have recently been admitted as privat-docents in German technical schools and universities: Dr. A. BARUCH, in mathematics, in the department of mines at the Berlin technical school; Dr. J. DEUXES, in mathematics, at the University of Göttingen; Dr. A. LANDE, in theoretical physics, at the University of Frankfurt a. M.

DR. A. CHATELET has been appointed professor of mathematics at the University of Lille.

DR. BOULIGAND has been appointed maître de conférences in mathematics at the University of Rennes.

AT Cambridge University, Mr. J. E. LITTLEWOOD, of Trinity College, has been appointed Cayley lecturer in mathematics, and Mr. J. H. GRACE, of Peterhouse, has been reappointed University lecturer in mathematics.

PROFESSOR L. E. DICKSON. of the University of Chicago, has been elected correspondent of the Paris academy of sciences, in the section of geometry, as successor to Professor E. COSSERAT, elected non-resident member.

AT the University of Texas, assistant professor R. L. MOORE, of the University of Pennsylvania, has been appointed associate professor of pure mathematics. Associate professor JESSIE M. JACOBS, of Rockland College, has been appointed instructor in pure mathematics and assistant professor J. N. MICHIE, of the Texas agricultural and mechanical college, has been appointed adjunct professor of applied mathematics.

YALE University has conferred the honorary degree of master of arts on Professor H. E. HAWKES, dean of Columbia College.

DR. DANIEL BUCHANAN, professor of astronomy and mathematics at Queen's University, Kingston, Ontario, has been appointed professor of mathematics and head of the department at the University of British Columbia. At Queen's University, assistant professor C. F. GUMMER has been promoted to an associate professorship, and Dr. NORMAN MILLER to an assistant professorship of mathematics; Mr. A. WOODS has resigned, to accept a lecturership at Western University, London, Ontario.

PROFESSOR S. LEFSCHETZ, of the University of Kansas, has been granted leave of absence for the year 1920–1921. He will spend the first term at the University of Paris, and the second at Padua and Bologna.

ASSISTANT professor E. G. BILL, of Dartmouth College, has been promoted to a full professorship of mathematics.

DR. J. W. CAMPBELL, of the State University of Iowa, has been appointed assistant professor of mathematics at the University of Alberta.

MISS GERTRUDE SMITH, of Vassar College, has been promoted to an assistant professorship of mathematics.

DR. J. R. KLINE, of the University of Illinois, has been appointed assistant professor of mathematics at the University of Pennsylvania.

DR. L. R. FORD, of Harvard University, has been appointed assistant professor of mathematics at Rice Institute.

AT Elmira College, Miss MARY SUFFA, of Beloit College, has been appointed head of the department of mathematics, and Miss FRANCIS W. WRIGHT, of Brown University, has been appointed instructor.

ASSISTANT professor A. C. MADDOX, of the Oklahoma Agricultural and Mechanical College, has been appointed professor of mathematics at the Louisiana State Normal School at Natchitoches.

MR. F. H. MURRAY and DR. J. L. WALSH have been appointed Sheldon Fellows by Harvard University, and expect to spend the coming year studying in Paris.

PROFESSOR IDA BARNEY, of Meredith College, has been appointed assistant professor of mathematics at Smith College.

ASSISTANT professor LOUISA M. WEBSTER, of Hunter College, has been promoted to an associate professorship of mathematics.

DR. C. A. LAISANT died May 5, 1920, at the age of seventy-nine years. He was founder of the *Intermédiare des Mathé*

maticiens, one of the founders of *l'Enseignement Mathématique*, and an editor of the *Nouvelles Annales de Mathématiques*.

MISS MARY A. COLPITTS, instructor in mathematics at the University of Wisconsin, died July 11, 1920.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AHRENS (W.). Mathematiker-Anekdoten. 2te, stark veränderte Auflage. (Mathematisch-physikalische Bibliothek, Nr. 18.) Leipzig, Teubner, 1920. 42 pp. M. 1.40

BARTHEL (E.). Polargeometrie. Berlin, Leonhard Simion Nachfolger, 1919. M. 4.50

CHINI (M.). Esercizi di calcolo infinitesimale. 3a edizione. Livorno, Giusti, 1920. 8vo. 10 + 300 pp. L. 8.50

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band II 3, Heft 3: L. Lichtenstein, Neuere Entwicklung der Potentialtheorie. Konforme Abbildung. Leipzig, Teubner, 1919. M. 7.60

FRICKE (R.). See PERRY (J.).

LICHTENSTEIN (L.). See ENCYKLOPÄDIE.

LUCKEY (P.). Einführung in die Nomographie. 2ter Teil: Die Zeichnung als Rechenmaschine. Leipzig, Teubner, 1920. M. 1.40

MEISSNER (O.). Wahrscheinlichkeitsrechnung. 1tes Bändchen: Grundlehren. 2tes Bändchen: Anwendungen. Leipzig, Teubner, 1919. M. 1.00 + 1.00

MUIR (T.). The theory of determinants in the historical order of development. Volume 3: The period 1861 to 1880. London, Macmillan, 1920. 26 + 503 pp. 35s.

PASCH (M.). Mathematik und Logik. 4 Abhandlungen. Leipzig, 1919. M. 2.00

PERRY (J.). Höhere Mathematik für Ingenieure. Autorisierte deutsche Bearbeitung von R. Fricke und F. Süchting. 3te Auflage. Leipzig, Teubner, 1919. M. 20.00

PIAGGIO (H. T. H.). An elementary treatise on differential equations and their applications. London, Bell, 1920. 12s.

PINCHERLE (S.). Lezioni di calcolo infinitesimale dettate nella R. Università di Bologna e redatte per uso degli studenti. 2a edizione riveduta. Bologna, Zanichelli, 1920. 8vo. 8 + 785 pp. L. 40.00

RIEMANN (B.). See WEBER (H.).

SMITH (W. W.). A theory of the mechanism of survival: the fourth dimension and its applications. London, Kegan Paul and Company, 1920. 196 pp. 5s.

STONEY (J.). Calculus for engineering students. New York, Pitman, 1920. 12mo. 140 pp. $1.35

SÜCHTING (F.). See PERRY (J.).

WEBER (H.). Die partiellen Differentialgleichungen der mathematischen Physik. Nach Riemanns Vorlesungen bearbeitet. 1ter Band. 6te Auflage. Braunschweig, 1919. M. 14.00

II. ELEMENTARY MATHEMATICS.

CRANTZ (P.). Arithmetik und Algebra zum Selbstunterricht. 2ter Teil: Gleichungen. 5te Auflage. Leipzig, 1919. M. 1.75

CRESPI (B. I.) e SOLUSTRI (A.). Esercizi e problemi di aritmetica, geometria, algebra, ad uso delle scuole medie inferiori. Roma, tip. del Genio civile, 1920. 8vo. 192 pp. L. 4.25

FROUMENTY (M.). See PHILIPPE (P.).

GERLACH (A.). Von schönen Rechenstunden. 4te vermehrte Auflage. Leipzig, Quelle und Meyer, 1919. Geb. M. 8.00

PANIZZA (F.). Aritmetica pratica. 4a edizione, riveduta e corretta. (Manuali Hoepli.) Milano, Hoepli, 1920. 24mo. 9 + 191 pp. L. 3.00

PHILIPPE (P.) et FROUMENTY (M.). Cours de géométrie. Tome 1. 2e édition. Paris, Dunod, 1920. 12mo. 8 + 272 pp. Fr. 9.75

SOLUSTRI (A.). See CRESPI (B. I.).

III. APPLIED MATHEMATICS.

ANGERSBACH (A.). Das Relativitätsprinzip leichtfasslich entwickelt. (Mathematisch-physikalische Bibliothek, Nr. 39.) Leipzig, Teubner, 1920. 57 pp.

BAIRSTOW (L.). Applied aerodynamics. London, Longmans, 1920. 12 + 566 pp. 32s.

BRENDEL (M.). See KLEIN (F.).

BROSE (H. L.). See SCHLICK (M.).

BROWN (E. W.). Tables of the motion of the moon. Prepared with the assistance of H. B. Hedrick. Six sections. New Haven, Yale University Press (printed by the Cambridge University Press), 1919. 13 + 140 + 39 + 223 + 99 + 56 + 102 pp. $20.00

DEPREZ (M.) et SOUBRIER (M.). Les lois fondamentales de l'électrotechnique. Paris, Dunod, 1920. 8vo. 758 pp. Cartonné. Fr. 34.50

DUPUIS (A.) et LOMBARD (J.). Cours de dessin industriel. Tome 3. 2e édition, nouveau tirage. Paris, Dunod, 1919. Fr. 9.00

EINSTEIN (A.). Ueber die spezielle und die allgemeine Relativitätstheorie. 4te Auflage. Braunschweig, 1919. M. 2.80

—— See LORENTZ (H. A.).

EWING (J. A.). Thermodynamics for engineers. Cambridge, University Press, 1920. 30s.

GAUSS (C. F.). See KLEIN (F.).

HEDRICK (H. B.). See BROWN (E. W.).

HENSELING (R.). Kleine Sternkunde. Stuttgart, Kosmos, Franckh, 1919. 109 pp. Geh. M. 3.60

HIRSCHAUER (L.). L'aviation de transport. L'évolution de la construction de 1907 à 1919 et la réalisation des avions de transport. L'utilisation économique des appareils. Paris, Dunod, 1919. 233 pp. + 3 cartes. Fr. 33.00

KLEIN (F.), BRENDEL (M.) und SCHLESINGER (L.). Materalien für eine wissenschaftliche Biographie von Gauss. Heft 7: Ueber die astronomischen Arbeiten von Gauss. 1ter Abschnitt: Theoretische Astronomie, von M. Brendel. Leipzig, Teubner, 1919. M. 6.00

LINDEMANN (F. A.). See SCHLICK (M.).

LOMBARD (J.). See DUPUIS (A.).

LORENTZ (H. A.), EINSTEIN (A.) und MINKOWSKI (H.). Das Relativitätsprinzip. 3te, verbesserte Auflage. Leipzig, Teubner, 1920. 146 pp.

MINKOWSKI (H.). See LORENTZ (H. A.).

NIEMÖLLER (M.). Zur Bewegung eines Punktes auf Rotationsflächen bei Wirkung eines auf der Rotationsachse gelegenen Anziehungszentrums. (Dissertation.) Halle, 1919.

RANELLETTI (C.). Elementi di geometria descrittiva, ad uso dei rr. istituti tecnici. 2a edizione. (Manuali Hoepli.) Milano, Hoepli, 1920. 24mo. 12 + 197 pp. con 2 tavole. L. 4.50

SCHEFFERS (G.). Lehrbuch der Mathematik für Studierende der Naturwissenschaften und der Technik. 4te verbesserte Auflage. Berlin, Vereinigung wissenschaftlicher Verleger (Walter de Gruyter), 1919. M. 42.00

SCHLESINGER (L.). See KLEIN (F.).

SCHLICK (M.). Space and time in contemporary physics. Rendered into English by H. L. Brose. Introduction by F. A. Lindemann. Oxford, University Press, 1920. 6s. 6d.

SOUBRIER (M.). See DEPREZ (M.).

THOMSON (G. P.). Applied aerodynamics. London, Hodder and Stoughton, 1920. 20 + 292 pp. 42s.

VALENTINER (S.). Die Grundlagen der Quantentheorie in elementarer Darstellung. 2te, erweiterte Auflage. Braunschweig, 1919. M. 3.60